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NORTH-HOLLAND

Tolerance Competition Graphs

R. C. Brigham

Department of Mathematics
University of Central Florida
Orlando, Florida 32816

F. R. McMorris*

Department of Mathematics
University of Louisville
Louisville, Kentucky 40292

and

R. P. Vitray

Department of Mathematical Science
Rollins College
Winter Park, Florida 32789

Dedicated to Professor John Maybee on the occasion of his 65th birthday.

Submitted by J. Richard Lundgren

ABSTRACT

The ϕ-tolerance competition graph is introduced as a generalization of the p-competition graphs defined by Kim, McKee, McMorris, and Roberts. Let ϕ be a symmetric function defined from $N \times N$ into N, where N denotes the nonnegative integers. $G = (V, E)$ is a ϕ-tolerance competition graph if there is a directed graph $D = (V, A)$ and an assignment of a nonnegative integer t_i to each vertex $v_i \in V$ such that, for $i \neq j$, $v_i v_j \in E(G)$ if and only if $|O(v_i) \cap$

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$O(v_j)| \geq \phi(t_i, t_j)$, where $O(x) = \{y : xy \in A\}$. A general characterization of ϕ-tolerance competition graphs is given, and specific results are obtained when ϕ is the minimum, maximum, and sum functions.

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1. INTRODUCTION

Suppose $D = (V, A)$ is a digraph. The *competition graph of* D is the graph $G = (V, E)$ where $xy \in E$ if and only if there exists $z \in V$ such that $xz, yz \in A$. Competition graphs have been studied extensively since being introduced in 1968 by J. Cohen [1–5, 10, 11, 13]. The original definition considered the situation where D represented the food web of an ecosystem and thus was usually assumed to be acyclic. However, competition graphs of arbitrary graphs have also been well studied. We impose no restrictions on a digraph unless otherwise specified. A graph G is a *competition graph* if it is the competition graph of some digraph.

There is another useful way to view competition graphs. First recall that a graph $G = (V, E)$ is the *intersection graph* of a family of sets $\{S_x : x \in V\}$ if $xy \in E$ with $x \neq y$ if and only if $S_x \cap S_y \neq \emptyset$. For a digraph $D = (V, A)$ and $x \in V$, let $O(x) = \{y : xy \in A\}$. Then the competition graph of D is simply the intersection graph of $\{O(x) : x \in V\}$. We now generalize this approach by requiring $|O(x) \cap O(y)|$ to be sufficiently large before introducing the edge xy in the competition graph. The most elementary example of this is the *p-competition graph* considered in [8], where xy is an edge of the p-competition graph if and only if $|O(x) \cap O(y)| \geq p$. Thus 1-competition graphs are the ordinary competition graphs defined above.

We now extend this even further to conform to the general tolerance intersection graph idea given in [6, 7]. Let ϕ be a symmetric function defined from $N \times N$ into N, where N denotes the nonnegative integers. Then $G = (V, E)$ is a *ϕ-tolerance competition graph* if there is a directed graph $D = (V, A)$ and an assignment of a nonnegative integer t_i to each vertex $v_i \in V$ such that, for $i \neq j$, $v_iv_j \in E(G)$ if and only if $|O(v_i) \cap O(v_j)| \geq \phi(t_i, t_j)$.

The following notion of covering the edges by cliques plays an important role in characterizing competition graphs. An *edge clique cover* of a graph G is a collection of complete subgraphs of G such that every edge is contained in at least one of these complete subgraphs. The *edge clique cover number* $\theta_e(G)$ [12] of G is the cardinality of a smallest edge clique cover of G.

Dutton and Brigham [2] showed that $G = (V, E)$ is a competition graph if and only if $\theta_e(G) \leq |V|$. Recently Kim, McKee, McMorris, and Roberts [8, 9] generalized these ideas. A *p-edge clique cover* (p-ECC) of G is a collection of subsets $S_1, S_2, \ldots, S_k$ of $V(G)$ such that the intersection of any p of the sets is complete, and the collection of all such p-intersections is an edge clique cover of G, i.e., $xy \in E(G)$ if and only if at least p of the sets S_i contain both x and y. The smallest value of k for which $S_1, S_2, \ldots, S_k$ is a p-ECC is the *p-edge clique cover number* of G and is denoted by $\theta_e^p(G)$. Kim et al. [8] show that G is a p-competition graph if and only if $\theta_e^p(G) \leq |V|$.

We generalize these definitions in a natural way. Let ϕ be a symmetric function defined from $N \times N$ into N, and $T = (t_1, t_2, \ldots, t_n)$ be an n-tuple of (not necessarily distinct) nonnegative integers. A *ϕ-T-edge clique cover* (ϕ-T–ECC) of a graph $G = (V, E)$ with vertices $v_1, v_2, \ldots, v_n$ is a collection $S_1, S_2, \ldots, S_k$ of subsets of V such that $v_r v_s \in E$ if and only if at least $\phi(t_r, t_s)$ of the sets S_i contain both v_r and v_s. The size k of a smallest ϕ-T-ECC of G taken over all vectors T is the *ϕ-T-edge clique cover number* and is denoted $\theta_\phi(G)$.

Here we will restrict ϕ to be one of the three functions minimum (min), maximum (max), and sum. The t_i will be referred to as *tolerances*, and T may be viewed as a function from the vertices of G to the nonnegative integers. The following theorem relates ϕ-tolerance competition graphs to the ϕ-T-ECC number in a way completely analogous to the p-competition graph situation.

THEOREM 1. *Let ϕ be a symmetric function defined from $N \times N$ into N. Then $G = (V, E)$ is a ϕ-tolerance competition graph if and only if $\theta_\phi(G) \leq |V|$.*

Proof. Suppose $G = (V, E)$ is a ϕ-tolerance competition graph. Let $D = (V, A)$ be a digraph giving rise to G, with $T = (t_1, t_2, \ldots, t_n)$ being the n-tuple of tolerances associated with V. For each $x \in V$, let $S_x = \{v : vx \in A\}$. Now $v_i v_j \in E(G)$ if and only if $|O(v_i) \cap O(v_j)| = |\{S_x : v_i, v_j \in S_x\}| \geq \phi(t_i, t_j)$, so G has a ϕ-T-ECC of size at most $|V|$. Now assume G has a ϕ-T-ECC of size at most $|V|$. Define a digraph $D = (V, A)$ by $A = \{v_i v_k : v_i \in S_k\}$. Then, in D, we have $|O(v_i) \cap O(v_j)| = |\{S_x : v_i, v_j \in S_x\}| \geq \phi(t_i, t_j)$ if and only if $v_i v_j \in E$, which means G is a ϕ-tolerance competition graph. ∎

In light of Theorem 1 and its proof, we will often show that a graph is a ϕ-tolerance competition graph by specifying the tolerances and sets of a ϕ-T-ECC of size at most $|V|$ without explicitly defining the digraph

D. Theorem 1 implies that any graph G can be turned into a ϕ-tolerance competition graph by adding at most $\theta_\phi(G) - |V|$ isolated vertices. The minimum number of such vertices needed represents an interesting parameter to study.

Theorem 1 is an extension of just one of several theorems which have become standard for competition graph concepts. The p-competition graph versions are stated in Kim et al. [8]. These characterize when a graph G is the p-competition graph of a loopless or acyclic digraph. All extend in a straightforward manner to ϕ-tolerance competition graphs, using analogous proofs. We list only one here.

THEOREM 2. *Suppose G is a graph on n vertices. Then G is a ϕ-tolerance competition graph of a loopless digraph if and only if G has a ϕ-T-ECC consisting of sets $S_1, S_2, \ldots, S_n$ and a labeling of vertices $v_1, v_2, \ldots, v_n$ such that $v_i \in S_j$ implies $i \neq j$.*

We have restricted the domain of ϕ to $N \times N$. That is not necessary, for tolerances could just as well be taken from the nonnegative reals. It is easy to see that this does not alter the set of min- and max-tolerance competition graphs, but it does affect the collection of sum-tolerance competition graphs, as we shall illustrate.

Although we will not make use of it in this paper, one can employ vector techniques to study ϕ-tolerance competition graphs. Let G be a ϕ-tolerance competition graph with vertices $v_1, v_2, \ldots, v_n$. Define an $n \times n$ matrix $A = (a_{i,j})$ by $a_{i,j} = 1$ and if $v_i \in O(v_j)$ and $a_{i,j} = 0$ otherwise. The rows of A correspond to the sets S_x defined in the proof of Theorem 1. Furthermore, if c_i and c_j are distinct columns of A, then v_i is adjacent to v_j in G if and only if the inner product $c_i \cdot c_j \geq \phi(t_i, t_j)$.

Sections 2, 3, and 4 discuss min-, max-, and sum-tolerance competition graphs, respectively. Section 5 concludes the paper with some open questions.

2. Min-TOLERANCE COMPETITION GRAPHS

A min-tolerance competition graph is a ϕ-tolerance competition graph in which ϕ is defined by the equation $\phi(t_i, t_j) = \min\{t_i, t_j\}$. In this section we show that several classes of graphs are min-tolerance competition graphs, and prove some sufficient condition theorems. We do not know of any necessary conditions for a graph to be a min-tolerance competition graph, and indeed, we have not discovered any graph which is not a min-tolerance competition graph.

THEOREM 3. *Every bipartite graph B is a* min-*tolerance competition graph.*

Proof. Let $\{a_1, a_2, \ldots, a_r\}$ and $\{b_1, b_2, \ldots, b_s\}$ be a bipartition of B. We describe a min-T-ECC of B. Let $t(a_i) = 1$ for $1 \le i \le r$ and $t(b_k) = r+1$ for $1 \le k \le s$. Let $S_i = \{a_i\} \cup N(a_i)$ for $1 \le i \le r$, where $N(v)$ is the set of vertices in B which are adjacent to the vertex v. It is easy to check that a_i and b_k are adjacent in B if and only if at least $\min(t(a_i), t(b_k))$ of the sets S_i contain both a_i and b_k. The size of the min-T-ECC is $r < |V|$, and thus B is a min-tolerance competition graph. ■

Note that the digraph $D = (V, A)$ which arises from the construction given in the proof of Theorem 3 will have a loop at each of the a_i's. This need not be the case. If $r \ge 2$, we define a digraph $D' = (V, A')$ where $a_i a_{i+1 \bmod r} \in A'$ for $1 \le i \le r$ and $b_j a_{i+1 \bmod r} \in A'$ for $1 \le i \le r$ and $1 \le j \le s$ if and only if $b_j a_i \in A$. If $r = 1$ and $s > 1$, reverse the roles of the a_i's and the b_j's to define D'. Finally, if $r = 1 = s$, let $A = \emptyset$ and assign tolerances of $1 - |E(B)|$ to both vertices. The implies the following.

OBSERVATION. *Every bipartite graph B is a min-tolerance-competition graph arising from a loopless digraph.*

An easy generalization of the proof of Theorem 3 leads to the following result.

THEOREM 4. *$K_{n_1, n_2, \ldots, n_k}$ is a* min-*tolerance competition graph.*

Proof. Let $K = K_{n_1, n_2, \ldots, n_k}$. Again we describe a min-T-ECC of size at most $|V(K)| = n_1 + n_2 + \cdots + n_k$. Let $V_1, V_2, \ldots, V_k$ be the independent sets of the k-partition of $V(K)$. For $v \in V_i$ assign $t(v) = 1 + n_1 + \cdots + n_{i-1}$ ($= 1$ if $i = 1$). For each vertex $v \in V_i$, where $1 \le i \le k-1$, let $S_v = \{v\} \cup \bigcup_{j=i+1}^{k} V_j$. Suppose $v \in V_i$ and $u \in V_j$ and, without loss of generality, $i \le j$. By construction, $\min(v, u) = t(v) = 1 + n_1 + \cdots + n_{i-1}$. If $i < j$, then u and v are both in S_v and in the $n_1 + n_2 + \cdots + n_{i-1}$ sets which correspond to the vertices in $V_1, V_2, \ldots, V_{i-1}$. If $i = j$, then the only sets containing both u and v are the $n_1 + n_2 + \cdots + n_{i-1}$ sets which correspond to the vertices in $V_1, V_2, \ldots, V_{i-1}$. Since $n_1 + n_2 + \cdots + n_{i-1} < \min(v, u) = 1 + n_1 + n_2 + \cdots + n_{i-1}$ and u is adjacent to v if and only if $i < j$, we have a min-T-ECC of size $n_1 + n_2 + \cdots + n_{k-1} < |V(K)|$. ■

In a min-tolerance competition graph, vertices which have tolerance 0 must be adjacent to every other vertex in the graph. It is not true, however, that such a vertex must always be assigned a tolerance of 0, as is shown in the next theorem.

THEOREM 5. *Let G be a* min-*tolerance competition graph with n vertices, and let M be the set of vertices in G of degree $n-1$. If $G-M$ has no isolated vertices, then G is a* min-*tolerance competition graph in which all tolerances can be assigned from the positive integers.*

Proof. Let the sets $S_1, S_2, \dots, S_j$ form a minimum T-ECC of G where $j \le n$. It is possible that at least one of the vertices in M has been assigned a tolerance of zero in this T-ECC. Now we create a minimum T-ECC $S'_1, S'_2, \dots, S'_j$ of $G-M$ where $S'_k = S_k - S_k \cap M$. Notice this uses the same tolerances for vertices of $G-M$ as were employed in the original T-ECC for G, and that these tolerances must all be greater than zero. Furthermore, each vertex of $G-M$ is listed in at least one set of this T-ECC of $G-M$, since $G-M$ has no isolated vertices, that is, every vertex in $G-M$ has a neighbor in G which is not in M and so must be in at least one set with that neighbor. We use these sets to create our desired minimum T-ECC for G. Let each vertex in M have tolerance 1, let all other vertices have their previously assigned tolerances (all greater than 0), and create sets $S'_1 \cup M, S'_2 \cup M, \dots S'_j \cup M$. These $j \le n$ sets form a minimum T-ECC of G in which all tolerances have been assigned from the positive integers. ■

The proof of the next theorem describes a construction of the sets and tolerances of a min-T-ECC which turns out to be useful in a number of examples. A *vertex cover* is a collection of vertices such that every edge is incident to at least one of the vertices in the collection.

THEOREM 6. *Let $S = \{s_1, s_2, \dots, s_k\}$ be a (not necessarily minimum) vertex cover of G, and $\langle S \rangle$ be the subgraph of G induced by S. Let G have order n. If $\theta_e^1 \langle S \rangle \le n-k$, then G is a* min-*tolerance competition graph. Moreover, the* min-*T-ECC has at most $k + \theta_e^1 \langle S \rangle$ sets, and only two tolerances are required.*

Proof. For each s_i, let $S_i = \{s_i\} \cup [N(s_i) - S]$. Let $C_1, C_2, \dots, C_r$ be the complete subgraphs of a minimum 1-ECC of $\langle S \rangle$, and define $S_{k+m} = C_m$ for $1 \le m \le r$. Since $C_1, C_2, \dots, C_r$ is a minimum 1-ECC of $\langle S \rangle$, we have $r = \theta_e^1 \langle S \rangle$; hence, we have $k + r = k + \theta_e^1 \langle S \rangle \le k + (n-k) = n$ sets. Let $t(s_1) = t(s_2) = \cdots = t(s_k) = 1$, and assign all other tolerances to be $n+1$. Let u and v be two vertices of G. If neither u nor v is in S, both of their tolerances are $n+1$, so they are not in $\min(t(u), t(v))$ sets. Also, they are not adjacent, since S is a vertex cover. Assume u is in S and v

is not. If they are not adjacent in G, then they are in none of the S_i's together. If u is adjacent to v, then $t(u) = 1$, and they are both in the set S_i where $u = s_i$. If u and v are both in S, then they are together in one of the sets derived from the C_m's if and only if they are adjacent. It follows that $\theta_{\min}(G) \leq n$. ■

For a graph G on n vertices let $\beta_0(G)$ be the independence number (i.e. the maximum size of an independent set of vertices), $\delta(G)$ be the minimum degree, and $\Delta(G)$ be the maximum degree. The next theorem relates these invariants to the property of being a min-tolerance competition graph.

THEOREM 7. *If $\beta_0(G) \geq n\Delta(G)/2[\delta(G)+1]$, then G is a* min-*tolerance competition graph.*

Proof. Let e be the number of edges of G, so $e \leq n\Delta(G)/2$. Let I be an independent set of vertices of G of size $\beta_0(G)$ and $W = V - I$. Since I is an independent set, W is a vertex cover of G. The number of edges which join a vertex of W to a vertex of I is at least $\beta_0(G)\delta(G)$, so the number of edges which join two vertices of W is at most

$$\begin{aligned} \frac{n\Delta(G)}{2} - \beta_0(G)\delta(G) &\leq \frac{n\Delta(G)}{2} - \frac{n\Delta(G)\delta(G)}{2[\delta(G)+1]} \\ &= \frac{n\Delta(G)}{2\,[\delta(G)+1]} \leq \beta_0(G) = |V - W|. \end{aligned}$$

Hence, $\theta_e^1(\langle W \rangle) \leq |V - W|$, and G is a min-tolerance competition graph by Theorem 6. ■

Theorem 7 tends not to be very useful unless $\delta(G)$ is fairly close to $\Delta(G)$. Nevertheless, it does have a couple of interesting (and easy) corollaries.

COROLLARY 7.1. *If G is regular and $\beta_0(G) \geq n\Delta(G)/2[\Delta(G)+1]$, then G is a* min-*tolerance competition graph.*

COROLLARY 7.2. *If $\delta(G) \geq \Delta(G) - 1$ and $\beta_0(G) \geq n/2$, then G is a* min-*tolerance competition graph.*

It should be noted that Corollary 7.1 actually says less than one might think, since if G is regular and the clique size $\omega(G) = 2$, then $\beta_0(G) \leq n/2$.

In light of Theorem 3, a natural next question would be whether every tripartite graph is a min-tolerance competition graph. We have been unable to settle this question in general, but have shown it under certain conditions.

THEOREM 8. *Let A, B, and C partition the vertices of a graph G into three independent sets where $|A| = r, |B| = s$, and $|C| = t$. For each $b \in B$ define $n_A(b)$ by*

$$n_A(b) = \min_{c \in C} |\{a \in A : a, b, c \textit{ define a triangle}\}|.$$

Then, if $t \geq rs - \sum_{b \in B} n_A(b)$, G is a min*-tolerance competition graph.*

Proof. Let $t(a) = 1$ for all $a \in A, t(b) = r + 1$ for all $b \in B$, and $t(c) = r + s + t + 1$ for all $c \in C$. For each $a \in A$, define $S_a = \{a\} \cup N(a)$. Then v is adjacent to vertex a in A if and only if v and a are together in S_a, since $\min(t(v), t(a)) = 1$. For each $b \in B$ create $r+1-n_A(b)$ sets containing b, and place each $c \in C$ which is adjacent to b in just enough of these sets so that, along with the S_a sets, b and c are together a total of $r+1$ times. This is possible, since b and c are together in at least $n_A(b)$ of the S_a sets. Hence, b and c are together in $\min(t(b), t(c)) = r+1$ sets. The total number of sets created is $r + \sum_{b \in B}[r + 1 - n_A(b)]$. Using the hypothesis of the theorem, we have $r + \sum_{b \in B}[r + 1 - n_A(b)] = r + rs + s - \sum_{b \in B} n_A(b) \leq r + s + t$. Once again the conclusion follows from Theorem 1. ■

COROLLARY 8.1. *If $|N(b) \cap N(c)| \geq r - 1$ for all $b \in B$ and for all $c \in C$, and $s \leq t$, then G is a* min*-tolerance competition graph.*

Proof. In this case $n_A(b) \geq r-1$ for all $b \in B$, so $\sum_{b \in B} n_A(b) \geq rs - s$, and we have $r + rs + s - \sum_{b \in B} n_A(b) \leq r + rs + s - rs + s \leq r + s + t$. ■

Note that all that is really needed for the corollary is $\sum_{b \in B} n_A(b) \geq rs - s$, which may be true even if $|N(b) \cap N(c)| < r - 1$ for some choices of b and c.

The next theorem is very easy to prove but is also quite useful.

THEOREM 9. *If there exists an embedding of G on a surface which has no more faces than vertices and in which two vertices are adjacent if and only if they have more than one face in common, then G is a* min*-tolerance competition graph.*

Proof. Assign a tolerance of 2 to every vertex, and let each set be the vertices incident to a particular face. ■

In view of the proof to Theorem 9, its conclusion could be restated as

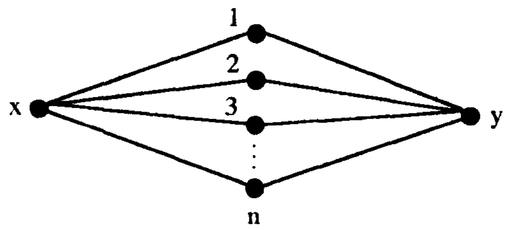

FIG. 1.

"G is a 2-competition graph."

3. Max-TOLERANCE COMPETITION GRAPHS

A max-tolerance competition graph is a ϕ-tolerance competition graph in which ϕ is defined by the equation $\phi(t_i, t_j) = \max\{t_i, t_j\}$. As in the min case, every p-competition graph is a max-tolerance competition graph using the single tolerance p. Unlike that case, however, not every bipartite graph is a max-tolerance competition graph. However, $K_{2,n}$ is a max-tolerance competition graph for all n.

THEOREM 10. *For all* $n \geq 1, K_{2,n}$ *is a* max-*tolerance competition graph. Moreover, this can be demonstrated using only tolerances* 0, 1, *and* 2.

Proof. Isaak, Kim, McKee, McMorris, and Roberts [5] show that $K_{2,n}$ is a 2-competition graph if and only if $n = 1$ or $n \geq 9$. A theorem of Dutton and Brigham [2] shows $K_{2,2}$ is a 1-competition graph. For $3 \leq n \leq 8$ we label $K_{2,n}$ as in Fig. 1 and assign tolerances by $t(y) = t(1) = 0, t(2) = 1$, $t(x) = t(3) = \cdots = t(n) = 2$. Then we demonstrate $K_{2,n}$ is a max-tolerance competition graph by listing the sets of a max-T-ECC. For $n = 3$, possible sets are $\{x, 1\}, \{x, 2\}$, $\{x, 1, 3\}, \{x, y, 2, 3\}$, and $\{3, y\}$; for $n = 4$, $\{x, 1, 3\}$, $\{x, 1, 4\}$, $\{x, y, 2, 3\}$, $\{x, 2, 4\}$, $\{y, 3, 4\}$, and $\{y, 4\}$; for $n = 5$, $\{x, 1, 3, 5\}$, $\{x, 1, 4\}$, $\{x, 2, 3\}$, $\{x, y, 2, 4, 5\}$, $\{y, 3, 4\}$, $\{y, 3\}$, and $\{y, 5\}$; for $n = 6$, $\{x, 1, 3, 5\}$, $\{x, 1, 4, 6\}$, $\{x, y, 2, 3, 6\}$, $\{x, 2, 4, 5\}$, $\{y, 3, 4\}$, $\{y, 5, 6\}$, $\{y, 4\}$, and $\{y, 5\}$; for $n = 7$, $\{x, y, 2, 3, 4, 5\}$, $\{x, 2, 6, 7\}$, $\{x, 1, 5\}$, $\{x, 1, 3, 6\}$, $\{x, 4, 7\}, \{y, 3, 7\}$, $\{y, 4, 6\}, \{y, 5, 7\}$, and $\{y, 6\}$; and for $n = 8$, $\{x, 1, 3, 6\}$, $\{x, 1, 4, 7\}$, $\{x, y, 2, 4, 6, 8\}$, $\{x, 2, 5, 7\}$, $\{x, 3, 5, 8\}$, $\{y, 3, 4\}$, $\{y, 5, 6\}$, $\{y, 7, 8\}, \{y, 3, 7\}$, and $\{y, 5\}$. ∎

The fact that not all bipartite graphs are max-tolerance competition graphs is illustrated by $K_{3,3}$. This has been demonstrated in a long multicase analysis based on the maximum tolerance used. We illustrate by showing a single subcase. Let the bipartite sets be $\{a, b, c\}$ and $\{x, y, z\}$. Let the subcase be defined as follows: the maximum tolerance of any vertex is 3, vertex x has tolerance 3, and x is in exactly four sets of a max-T-ECC (it must be in at least four). Then each of a, b, and c must be in three of the four sets, and any two of them must share at most two sets, forcing the following sets: $\{x, a, b, c\}, \{x, a, b\}, \{x, a, c\}$, and $\{x, b, c\}$. This means at least two of a, b, and c, say a and b, must have tolerance 3. Thus y and z must be in at least three sets with each of a and b, at most two sets with x, and at most two sets together; and no two of a, b, and c can be in another set together. These restrictions make it impossible to create a max-T-ECC using only six sets, and we conclude this case cannot occur. We must then test for x being in five sets and six sets, and for all other values for the maximum tolerance. All cases fail, and the desired conclusion is reached.

4. Sum-TOLERANCE COMPETITION GRAPHS

In this section ϕ is defined by the equation $\phi(t_i, t_j) = t_i + t_j$. Unlike the case of min- and max-tolerance competition graphs, restricting the tolerance of vertices to integers does make a difference, as is clear from the next two theorems.

THEOREM 11. *The graph $K_{3,3}$ is a sum-tolerance competition graph if noninteger tolerances are allowed.*

Proof. Let $\{a, b, c\}$ and $\{x, y, z\}$ be a partition of the vertices of $K_{3,3}$ into two independent sets. Assign $t(a) = t(b) = \frac{1}{3}$, $t(c) = \frac{3}{4}, t(x) = t(y) = \frac{1}{4}$, and $t(z) = 1$. Let $S_1 = \{a, c, x, z\}$, $S_2 = \{b, c, y, z\}, S_3 = \{a, y\}, S_4 = \{a, z\}, S_5 = \{b, z\}$, and $S_6 = \{b, x\}$. It is easy to see that two vertices u and v are adjacent if and only if they are together in $t(u) + t(v)$ of the S_i's. ■

THEOREM 12. *$K_{3,3}$ is not a sum-tolerance competition graph if tolerances are restricted to the integers.*

Proof. Once again let $\{a, b, c\}$ and $\{x, y, z\}$ be a partition of the vertices of $K_{3,3}$ into two independent sets. Suppose that $K_{3,3}$ is a sum-tolerance competition graph with tolerances restricted to the integers. By Theorem 1, there is a sum-T-ECC for $K_{3,3}$ whose size is at most six. Without loss of generality, let $t(x)$ be the maximum tolerance and $t(a) \leq t(b) \leq t(c)$. For any subset S of the vertices, define $[S]$ to be the number of sets in the sum-T-ECC which contain S as a subset, and (S) to be the number of sets in the sum-T-ECC whose intersection with S is nonempty. Note, for example, that if $S_1 \supseteq \{a,b\}$ and $S_2 \supseteq \{x,a\}$, then $[S_1] \le t(a)+t(b)-1$ and $[S_2] \ge t(x)+t(a)$. Similar inequalities apply to every other pair of vertices in the graph. Also, $[\{x,a,b\}] \le t(a)+t(b)-1$ and $[\{x,a,b,c\}] \ge 0$. Using elementary properties of set theory, we have

$$\begin{aligned}[\{x\}] \ \geq \ & [\{x,a\}] + [\{x,b\}] + [\{x,c\}] - [\{x,a,b\}] \\ & - [\{x,a,c\}] - [\{x,b,c\}] + [\{x,a,b,c\}].\end{aligned}$$

Hence

$$[\{x\}] \ge 3t(x) + 3 - t(a) - t(b) - t(c). \qquad (*)$$

This inequality proves to be very useful as we argue by cases depending on the value of $t(x)$. We shall do only the first two cases here, as the rest are similarly straightforward.

Case 1: $t(x) = 0$. All tolerances would have to be zero, and the graph would have to be complete.

Case 2: $t(x) = 1$. All tolerances are either 0 or 1, and there are at most two vertices of tolerance 0 (more would create unwanted edges).

Case 2a: No vertices have tolerance 0. Then $K_{3,3}$ would be a 2-competition graph, which it is not [5].

Case 2b: Either one or two vertices have tolerance 0. Without loss of generality we may assume $t(a) = 0$ and $t(b) = t(c) = 1$. Also, y and z cannot both have tolerance 0, so without loss of generality $t(y) = 1$. By $(*)$, $[\{x\}] \ge 4$, and similarly $[\{y\}] \ge 4$. Now $(\{x,y\}) = [\{x\}] + [\{y\}] - [\{x,y\}]$. Since x and y are not adjacent, $[\{x,y\}] \le 1$, and therefore $(\{x,y\}) \ge 7$, which contradicts $n \le 6$. ■

5. OPEN QUESTIONS

There are several obvious questions which require further study. We list a few here.

(1) Are there graphs which are not min-tolerance competition graphs?
(2) Even more restrictive, are there tripartite graphs which are not min-tolerance competition graphs? In particular, is any subgraph of $K_{3,3,3}$ not a min-tolerance competition graph?

(3) Are there graphs which are not sum-tolerance competition graphs if noninteger tolerances are allowed?
(4) Analogous definitions can be made in which the desired clique cover is of the vertices rather than the edges. What can be determined in such cases?

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